

# First and second cohomologies of grading-restricted vertex algebras

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## Abstract

Let  $V$  be a grading-restricted vertex algebra and  $W$  a  $V$ -module. We show that for any  $m \in \mathbb{Z}_+$ , the first cohomology  $H_m^1(V, W)$  of  $V$  with coefficients in  $W$  introduced by the author is linearly isomorphic to the space of derivations from  $V$  to  $W$ . In particular,  $H_m^1(V, W)$  for  $m \in \mathbb{N}$  are equal (and can be denoted using the same notation  $H^1(V, W)$ ). We also show that the second cohomology  $H_{\frac{1}{2}}^2(V, W)$  of  $V$  with coefficients in  $W$  introduced by the author corresponds bijectively to the set of equivalence classes of square-zero extensions of  $V$  by  $W$ . In the case that  $W = V$ , we show that the second cohomology  $H_{\frac{1}{2}}^2(V, V)$  corresponds bijectively to the set of equivalence classes of first order deformations of  $V$ .

## 1 Introduction

The present paper is a sequel to the paper [H]. We discuss the first and second cohomologies of grading-restricted vertex algebras introduced by the author in that paper.

Let  $V$  be a grading-restricted vertex algebra and  $W$  a  $V$ -module. Recall from [H] that for each  $m \in \mathbb{Z}_+$  and  $n \in \mathbb{N}$ , we have an  $n$ -th cohomology  $H_m^n(V, W)$  of  $V$  with coefficients in  $W$ . For each  $n \in \mathbb{N}$ , We also have an  $n$ -th cohomology  $H_\infty^n(V, W)$  of  $V$  with coefficients in  $W$  which is isomorphic to the inverse limit of the inverse system  $\{H_m^n(V, W)\}_{m \in \mathbb{Z}_+}$ . We also have an additional second cohomology  $H_{\frac{1}{2}}^2(V, W)$  of  $V$  with coefficients in  $W$ . In the present paper, we discuss only  $H_m^1(V, W)$  for  $m \in \mathbb{Z}_+$  and  $H_{\frac{1}{2}}^2(V, W)$ .

Let  $V$  be a grading-restricted vertex algebra and  $W$  a  $V$ -module. A grading-preserving linear map  $f : V \rightarrow W$  is called a *derivation* if

$$\begin{aligned} f(Y_V(u, z)v) &= Y_{WV}^W(f(u), z)v + Y_W(u, z)f(v) \\ &= e^{zL(-1)}Y_W(v, -z)f(u) + Y_W(u, z)f(v) \end{aligned}$$

for  $u, v \in V$ . We use  $\text{Der}(V, W)$  to denote the space of all such derivations. We have the following result for the first cohomologies of  $V$  with coefficients in  $W$ :

**Theorem 1.1.** *Let  $V$  be a grading-restricted vertex algebra and  $W$  a  $V$ -module. Then  $H_m^1(V, W)$  is linearly isomorphic to the space of derivations from  $V$  to  $W$  for any  $m \in \mathbb{Z}_+$ , that is,  $H_m^1(V, W)$  is linearly isomorphic to  $\text{Der}(V, W)$  for any  $m \in \mathbb{Z}_+$ .*

In particular,  $H_m^1(V, W)$  for  $m \in \mathbb{N}$  are isomorphic (and can be denoted using the same notation  $H^1(V, W)$ ).

**Definition 1.2.** Let  $V$  be a grading-restricted vertex algebra. A *square-zero ideal* of  $V$  is an ideal  $W$  of  $V$  such that for any  $u, v \in W$ ,  $Y_V(u, x)v = 0$ .

**Definition 1.3.** Let  $V$  be a grading-restricted vertex algebra and  $W$  a  $\mathbb{Z}$ -graded  $V$ -module. A *square-zero extension*  $(\Lambda, f, g)$  of  $V$  by  $W$  is a grading-restricted vertex algebra  $\Lambda$  together with a surjective homomorphism  $f : \Lambda \rightarrow V$  of grading-restricted vertex algebras such that  $\ker f$  is a square-zero ideal of  $\Lambda$  (and therefore a  $V$ -module) and an injective homomorphism  $g$  of  $V$ -modules from  $W$  to  $\Lambda$  such that  $g(W) = \ker f$ . Two square-zero extensions  $(\Lambda_1, f_1, g_1)$  and  $(\Lambda_2, f_2, g_2)$  of  $V$  by  $W$  are *equivalent* if there exists an isomorphism of grading-restricted vertex algebras  $h : \Lambda_1 \rightarrow \Lambda_2$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \xrightarrow{g_1} & \Lambda_1 & \xrightarrow{f_1} & V \longrightarrow 0 \\ & & \downarrow 1_W & & \downarrow h & & \downarrow 1_V \\ 0 & \longrightarrow & W & \xrightarrow{g_2} & \Lambda_2 & \xrightarrow{f_2} & V \longrightarrow 0, \end{array}$$

is commutative.

The notion of square-zero extension of  $V$  by  $W$  is an analogue of the notion of square-zero extension of an associative algebra by a bimodule. (see, for example, Section 9.3 of [W]).

We have the following result for the second cohomology  $H_{\frac{1}{2}}^2(V, W)$  of  $V$  with coefficients in  $W$ :

**Theorem 1.4.** *Let  $V$  be a grading-restricted vertex algebra and  $W$  a  $V$ -module. Then the set of the equivalence classes of square-zero extensions of  $V$  by  $W$  corresponds bijectively to  $H_{\frac{1}{2}}^2(V, W)$ .*

**Definition 1.5.** Let  $t$  be a complex variable. A family of grading-restricted vertex algebras up to the first order in  $t$  is a  $\mathbb{Z}$ -graded vector space  $V$ , a family  $Y_t : V \otimes V \rightarrow V((x))$  for  $t \in \mathbb{C}$  of linear maps of the form  $Y_t = Y_0 + t\Psi$  where  $Y_0$  and  $\Psi$  are linear maps from  $V \otimes V$  to  $V((x))$  independent of  $t$ , and an element  $\mathbf{1} \in V$  such that  $(V, Y_t, \mathbf{1})$  satisfies all the axioms for grading-restricted vertex algebras up to the first order in  $t$ .

**Definition 1.6.** Let  $(V, Y_V, \mathbf{1})$  be a grading-restricted vertex algebra. A first order deformation of  $V$  is a family  $Y_t : V \otimes V \rightarrow V((x))$  for  $t \in \mathbb{C}$  of linear maps of the form  $Y_t = Y_V + t\Psi$  where

$$\begin{aligned}\Psi : V \otimes V &\rightarrow V((x)) \\ v_1 \otimes v_2 &\rightarrow \Psi(v_1, x)v_2\end{aligned}$$

is a linear map such that  $(V, Y_t, \mathbf{1})$  for  $t \in \mathbb{C}$  is a family of grading-restricted vertex algebras up to the first order in  $t$ . Two first order deformations  $Y_t^{(1)}$  and  $Y_t^{(2)}$ ,  $t \in \mathbb{C}$ , of  $(V, Y_V, \mathbf{1})$  are *equivalent* if there exists a family  $f_t : V \rightarrow V$ ,  $t \in \mathbb{C}$ , of linear maps of the form  $f_t = 1_V + tg$  where  $g : V \rightarrow V$  is a linear map preserving the gradings of  $V$  such that

$$f_t(Y_t^{(1)}(v_1, x)v_2) - Y_t^{(1)}(f_t(v_1), x)f_t(v_2) \in t^2V((x)) \quad (1.1)$$

for  $v_1, v_2 \in V$ .

We have:

**Theorem 1.7.** *The set of equivalence classes of first order deformations of a grading-restricted vertex algebra is in bijection with the set of equivalence classes of square-zero extensions of  $V$  by  $V$ .*

From Theorems 1.4 and 1.8, we obtain immediately the following result for the second cohomology  $H_{\frac{1}{2}}^2(V, V)$  of  $V$  with coefficients in  $V$ :

**Theorem 1.8.** *Let  $V$  be a grading-restricted vertex algebra. Then the set of the equivalence classes of first order deformations of  $V$  correspond bijectively to  $H_{\frac{1}{2}}^2(V, V)$ .*

We prove Theorems 1.1, 1.4 and 1.7 in Sections 2, 3 and 4, respectively.

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## 2 First cohomologies and spaces of derivations

We prove Theorem 1.1 in the present section. First, we need the following:

**Lemma 2.1.** *Let  $f : V \rightarrow W$  be a derivation. Then  $f(\mathbf{1}) = 0$ .*

*Proof.* By definition,

$$\begin{aligned} f(\mathbf{1}) &= f(Y_V(\mathbf{1}, z)\mathbf{1}) \\ &= \lim_{z \rightarrow 0} f(Y_V(\mathbf{1}, z)\mathbf{1}) \\ &= \lim_{z \rightarrow 0} e^{zL(-1)} Y_W(\mathbf{1}, -z) f(\mathbf{1}) + \lim_{z \rightarrow 0} Y_W(\mathbf{1}, z) f(\mathbf{1}) \\ &= 2f(\mathbf{1}). \end{aligned}$$

So  $f(\mathbf{1}) = 0$ . ■

Let  $\Phi : V \rightarrow \widetilde{W}_{z_1}$  be an element of  $C_m^1(V, W)$  satisfying  $\delta_m^1 \Phi = 0$ . Since  $\Phi$  satisfies the  $L(0)$ -conjugation property, for  $v \in V_{(n)}$  and  $z \in \mathbb{C}^\times$ ,

$$\begin{aligned} z^{L(0)}(\Phi(v))(0) &= (\Phi(z^{L(0)}v))(0) \\ &= z^n(\Phi(v))(0). \end{aligned}$$

Thus  $(\Phi(v))(0) \in W_{(n)}$ . So  $(\Phi(v))(0)$  is a grading-preserving linear map from  $V$  to  $W$ .

Since  $\delta_m^1 \Phi = 0$ ,

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(z_2) \rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1 - z_2)v_2))(z_2) \rangle) \\ &\quad + R(\langle w', Y_W(v_2, z_2)(\Phi(v_1))(z_1) \rangle) \\ &= 0 \end{aligned}$$

for  $v_1, v_2 \in V$  and  $w' \in W'$ . By  $L(-1)$ -derivative property for  $\Phi$  and the vertex operator map  $Y_W$ ,

$$R(\langle w', Y_W(v_2, z_2)(\Phi(v_1))(z_1) \rangle) = R(\langle w', e^{z_1 L(-1)} Y_W(v_2, -z_1 + z_2)(\Phi(v_1))(0) \rangle).$$

Thus we have

$$\begin{aligned} & R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(z_2) \rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1 - z_2)v_2))(z_2) \rangle) \\ & \quad + R(\langle w', e^{z_1 L(-1)} Y_W(v_2, -z_1 + z_2)(\Phi(v_1))(0) \rangle) \\ & = 0. \end{aligned}$$

Let  $z_2 = 0$ . We obtain

$$\begin{aligned} & R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(0) \rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1)v_2))(0) \rangle) \\ & \quad + R(\langle w', e^{z_1 L(-1)} Y_W(v_2, -z_1)(\Phi(v_1))(0) \rangle) \\ & = 0. \end{aligned}$$

Since  $w'$  is arbitrary, we obtain

$$\begin{aligned} & (\Phi(Y_V(v_1, z_1)v_2))(0) \\ & = e^{z_1 L(-1)} Y_W(v_2, -z_1)(\Phi(v_1))(0) + Y_W(v_1, z_1)(\Phi(v_2))(0) \\ & = Y_{WV}^W((\Phi(v_1))(0), z_1)(\Phi(v_2))(0) + Y_W(v_1, z_1)(\Phi(v_2))(0) \end{aligned}$$

for  $v_1, v_2 \in V$ . This means that  $(\Phi(\cdot))(0) : V \rightarrow W$  is a derivation from  $V$  to  $W$ . Note that  $\delta_m^0(C_m^0(V, W)) = 0$ . So we obtain a linear map from  $H^1(V, W)$  to the space of derivations from  $V$  to  $W$ .

Conversely, given any derivation  $f$  from  $V$  to  $W$ , let  $\Phi_f : V \rightarrow \widetilde{W}_{z_1}$  be given by

$$(\Phi_f(v))(z_1) = f(Y_V(v, z_1)\mathbf{1}) = Y_{WV}^W(f(v), z_1)\mathbf{1}$$

for  $v \in V$ , where we have used Lemma 2.1. By Theorem 5.6.2 in [FHL], the map from  $V$  to  $\widetilde{W}_{z_1}$  given by  $v \mapsto Y_{WV}^W((\Phi(v))(0), z_1)\mathbf{1}$  is composable with  $m$  vertex operators for any  $m \in \mathbb{N}$ . Thus  $\Phi_f \in C_m^1(V, W)$  for any  $m \in \mathbb{N}$ . For  $v_1, v_2 \in V$  and  $w' \in W'$ ,

$$\begin{aligned} & ((\delta_m^1 \Phi_f)(v_1 \otimes v_2))(z_1, z_2) \\ & = R(\langle w', Y_W(v_1, z_1)Y_{WV}^W(f(v_2), z_2)\mathbf{1} \rangle) \\ & \quad - R(\langle w', Y_{WV}^W(f(Y_V(v_1, z_1 - z_2)v_2), z_2)\mathbf{1} \rangle) \\ & \quad + R(\langle w', Y_W(v_2, z_2)Y_{WV}^W(f(v_1), z_1)\mathbf{1} \rangle) \\ & = R(\langle w', Y_W(v_1, z_1)Y_{WV}^W(f(v_2), z_2)\mathbf{1} \rangle) \\ & \quad - R(\langle w', Y_{WV}^W(Y_{WV}^W(f(v_1), z_1 - z_2)v_2), z_2)\mathbf{1} \rangle) \\ & \quad - R(\langle w', Y_{WV}^W(Y_W(v_1, z_1 - z_2)f(v_2), z_2)\mathbf{1} \rangle) \end{aligned}$$

$$\begin{aligned}
& +R(\langle w', Y_W(v_2, z_2)Y_{WV}^W(f(v_1), z_1)\mathbf{1} \rangle) \\
& = R(\langle w', Y_W(v_1, z_1)Y_{WV}^W(f(v_2), z_2)\mathbf{1} \rangle) \\
& \quad -R(\langle w', e^{z_2L_W(-1)}Y_{WV}^W(f(v_1), z_1 - z_2)v_2 \rangle) \\
& \quad -R(\langle w', e^{z_2L_W(-1)}Y_W(v_1, z_1 - z_2)f(v_2) \rangle) \\
& \quad +R(\langle w', Y_W(v_2, z_2)Y_{WV}^W(f(v_1), z_1)\mathbf{1} \rangle) \\
& = R(\langle w', Y_W(v_1, z_1)Y_{WV}^W(f(v_2), z_2)\mathbf{1} \rangle) \\
& \quad -R(\langle w', Y_{WV}^W(f(v_1), z_1)e^{z_2L_V(-1)}v_2 \rangle) \\
& \quad -R(\langle w', Y_W(v_1, z_1)e^{z_2L_W(-1)}f(v_2) \rangle) \\
& \quad +R(\langle w', Y_W(v_2, z_2)Y_{WV}^W(f(v_1), z_1)\mathbf{1} \rangle) \\
& = R(\langle w', Y_W(v_1, z_1)Y_{WV}^W(f(v_2), z_2)\mathbf{1} \rangle) \\
& \quad -R(\langle w', Y_{WV}^W(f(v_1), z_1)Y_W(v_2, z_2)\mathbf{1} \rangle) \\
& \quad -R(\langle w', Y_W(v_1, z_1)Y_{WV}^W(f(v_2), z_2)\mathbf{1} \rangle) \\
& \quad +R(\langle w', Y_W(v_2, z_2)Y_{WV}^W(f(v_1), z_1)\mathbf{1} \rangle) \\
& = -R(\langle w', Y_{WV}^W(f(v_1), z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
& \quad +R(\langle w', Y_W(v_2, z_2)Y_{WV}^W(f(v_1), z_1)\mathbf{1} \rangle).
\end{aligned} \tag{2.1}$$

From Theorem 5.6.2 in [FHL], we know that the right-hand side of (2.1) is 0. So we obtain a linear map  $f \mapsto \Phi_f$  from the space  $\text{Der}(V, W)$  to  $H_m^1(V, W) = C_m^1(V, W)$ .

Clearly these two maps are inverse to each other and thus  $\text{Der}(V, W)$  and  $H_m^1(V, W)$  are isomorphic.  $\blacksquare$

### 3 Second cohomologies and square-zero extensions

In this section, we prove Theorem 1.4.

Let  $(\Lambda, f, g)$  be a square-zero extension of  $V$  by  $W$ . Then there is an injective linear map  $\Gamma : V \rightarrow \Lambda$  such that the linear map  $h : V \oplus W \rightarrow \Lambda$  given by  $h(v, w) = \Gamma(v) + g(w)$  is a linear isomorphism. By definition, the restriction of  $h$  to  $W$  is the isomorphism  $g$  from  $W$  to  $\ker f$ . Then the grading-restricted vertex algebra structure and the  $V$ -module structure on  $\Lambda$  give a grading-restricted vertex algebra structure and a  $V$ -module structure

on  $V \oplus W$  such that the embedding  $i_2 : W \rightarrow V \oplus W$  and the projection  $p_1 : V \oplus W \rightarrow V$  are homomorphisms of grading-restricted vertex algebras. Moreover,  $\ker p_1$  is a square-zero ideal of  $V \oplus W$ ,  $i_2$  is an injective homomorphism such that  $i_2(W) = \ker p_1$  and the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V \longrightarrow 0 \\
& & \downarrow 1_W & & \downarrow h & & \downarrow 1_V \\
0 & \longrightarrow & W & \xrightarrow[g]{} & \Lambda & \xrightarrow[f]{} & V \longrightarrow 0
\end{array} \tag{3.1}$$

of  $V$ -modules is commutative. So we obtain a square-zero extension  $(V \oplus W, p_1, i_2)$  equivalent to  $(\Lambda, f, g)$ . We need only consider square-zero extension of  $V$  by  $W$  of the particular form  $(V \oplus W, p_1, i_2)$ . Note that the difference between two such square-zero extensions are in the vertex operator maps. So we use  $(V \oplus W, Y_{V \oplus W}, p_1, i_2)$  to denote such a square-zero extension.

We now write down the vertex operator map for  $V \oplus W$  explicitly. Since  $(V \oplus W, Y_{V \oplus W}, p_1, i_2)$  is a square-zero extension of  $V$ , there exists  $\Psi(u, x)v \in W((x))$  for  $u, v \in V$  such that

$$\begin{aligned}
Y_{V \oplus W}((v_1, 0), x)(v_2, 0) &= (Y_V(v_1, x)v_2, \Psi(v_1, x)v_2), \\
Y_{V \oplus W}((v_1, 0), x)(0, w) &= (0, Y_V(v_1, x)w_2), \\
Y_{V \oplus W}((0, w_1), x)(v_2, 0) &= (0, Y_{WV}^W(w, x)v_2), \\
Y_{V \oplus W}((0, w_1), x)(0, w_2) &= 0
\end{aligned}$$

for  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ . Thus we have

$$\begin{aligned}
Y_{V \oplus W}((v_1, w_1), x)(v_2, w_2) \\
= (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi(v_1, x)v_2)
\end{aligned} \tag{3.2}$$

for  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ .

The vacuum of  $V \oplus W$  is  $(\mathbf{1}, 0)$ . Since

$$\begin{aligned}
Y_{V \oplus W}((v, w), x)(\mathbf{1}, 0) &= e^{xL_{V \oplus W}(-1)}(v, w) \\
&= (e^{xL_V(-1)}v, e^{xL_W(-1)}w) \\
&= (Y_V(v, x)\mathbf{1}, Y_{WV}^W(w, x)\mathbf{1})
\end{aligned}$$

for  $v \in V$  and  $w \in W$ , we have

$$\Psi(v, x)\mathbf{1} = 0 \tag{3.3}$$

for  $v \in V$ .

We identify  $(V \oplus W)'$  with  $V' \oplus W'$ . For  $v_1, v_2 \in V$  and  $w' \in W'$ ,

$$\begin{aligned}
& \langle (0, w'), Y_{V \oplus W}((v_1, 0), z_1) Y_{V \oplus W}((v_2, 0), z_2) (\mathbf{1}, 0) \rangle \\
&= \langle w', \Psi(v_1, z_1) Y_V(v_2, z_2) \mathbf{1} + Y_W(v_1, z_1) \Psi(v_2, z_2) \mathbf{1} \rangle \\
&= \langle w', \Psi(v_1, z_1) Y_V(v_2, z_2) \mathbf{1} \rangle, \\
& \langle (0, w'), Y_{V \oplus W}((v_2, 0), z_2) Y_{V \oplus W}((v_1, 0), z_1) (\mathbf{1}, 0) \rangle \\
&= \langle w', \Psi(v_2, z_2) Y_V(v_1, z_1) \mathbf{1} + Y_W(v_2, z_2) \Psi(v_1, z_1) \mathbf{1} \rangle \\
&= \langle w', \Psi(v_2, z_2) Y_V(v_1, z_1) \mathbf{1} \rangle, \\
& \langle (0, w'), Y_{V \oplus W}(Y_{V \oplus W}((v_1, 0), z_1 - z_2)(v_2, 0), z_2) (\mathbf{1}, 0) \rangle \\
&= \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2) \mathbf{1} + \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2) \mathbf{1} \rangle \\
&= \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2) \mathbf{1} \rangle
\end{aligned}$$

are absolutely convergent in the region  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$ ,  $|z_2| > |z_1 - z_2| > 0$ , respectively, to one rational function in  $z_1$  and  $z_2$  with the only possible poles at  $z_1, z_2 = 0$  and  $z_1 = z_2$ . Using our notation in [H], we denote this rational function by

$$R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2) \mathbf{1} \rangle)$$

or

$$R(\langle w', \Psi(v_2, z_2) Y_V(v_1, z_1) \mathbf{1} \rangle)$$

or

$$R(\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2) \mathbf{1} \rangle).$$

Then we obtain an element, denoted by

$$E(\Psi(v_1, z_1) Y_V(v_2, z_2) \mathbf{1})$$

or

$$E(\Psi(v_2, z_2) Y_V(v_1, z_1) \mathbf{1})$$

or

$$E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2) \mathbf{1}),$$

of  $\widetilde{W}_{z_1, z_2}$  given by

$$\langle w', E(\Psi(v_1, z_1) Y_V(v_2, z_2) \mathbf{1}) \rangle = R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2) \mathbf{1} \rangle)$$



or

$$\langle w', E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \rangle = R(\langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1} \rangle)$$

or

$$\langle w', E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}) \rangle = R(\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1} \rangle).$$

By definition, we have

$$\begin{aligned} E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \\ &= E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}) \end{aligned}$$

for  $v_1, v_2 \in V$ .

Let

$$\Phi : V \otimes V \rightarrow \widetilde{W}_{z_1, z_2}$$

be the linear map given by

$$\begin{aligned} (\Phi(v_1 \otimes v_2))(z_1, z_2) &= E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \\ &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \\ &= E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}) \end{aligned} \quad (3.4)$$

for  $v_1, v_2 \in V$  and  $(z_1, z_2) \in F_2\mathbb{C}$ . We first prove that  $\Phi \in \widehat{C}_{\frac{1}{2}}^2(V, W)$ .

By the  $L(-1)$ -derivative property and the  $L(0)$ -bracket formula for  $V \oplus W$ , we have

$$\frac{d}{dx}Y_{V \oplus W}((v, 0), x) = Y_{V \oplus W}((L_V(-1)v, 0), x) \quad (3.5)$$

$$= [L_{V+W}(-1), Y_{V \oplus W}((v, 0), x)], \quad (3.6)$$

$$\begin{aligned} [L_{V+W}(0), Y_{V \oplus W}((v, 0), x)] &= Y_{V \oplus W}((L_V(0)v, 0), x) + x \frac{d}{dx}Y_{V \oplus W}((v, 0), x) \end{aligned} \quad (3.7)$$

for  $v \in V$ . By (3.5), (3.6), (3.7), (3.2) and the  $L(-1)$ -derivative property and the  $L(0)$ -bracket formula for  $V$ , we obtain

$$\frac{d}{dx}\Psi(v, x) = \Psi(L_V(-1)v, x) \quad (3.8)$$

$$= L_W(-1)\Psi(v, x) - \Psi(v, x)L_V(-1), \quad (3.9)$$

$$L_W(0)\Psi(v, x) - \Psi(v, x)L_V(0) = \Psi(L_V(0)v, x) + x \frac{d}{dx}\Psi(v, x) \quad (3.10)$$

for  $v \in V$ . From (3.10), we obtain

$$z^{L_W(0)}\Psi(v, x) = \Psi(z^{L_V(0)}v, zx)z^{L_V(0)} \quad (3.11)$$

for  $v \in V$ .

For  $v_1, v_2 \in V$  and  $w' \in W'$ , by (3.8) and the  $L(-1)$ -derivative property for  $V$ , we obtain

$$\begin{aligned} & \frac{\partial}{\partial z_1} \langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2) \rangle \\ &= \frac{\partial}{\partial z_1} \langle w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\ &= \frac{\partial}{\partial z_1} R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\ &= R\left(\left\langle w', \frac{\partial}{\partial z_1} \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \right\rangle\right) \\ &= R(\langle w', \Psi(L_V(-1)v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\ &= \langle w', E(\Psi(L_V(-1)v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\ &= \langle w', (\Phi(L_V(-1)v_1 \otimes v_2))(z_1, z_2) \rangle \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} & \frac{\partial}{\partial z_2} \langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2) \rangle \\ &= \frac{\partial}{\partial z_2} \langle w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\ &= \frac{\partial}{\partial z_2} R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\ &= R\left(\left\langle w', \Psi(v_1, z_1) \frac{\partial}{\partial z_2} Y_V(v_2, z_2)\mathbf{1} \right\rangle\right) \\ &= R(\langle w', \Psi(v_1, z_1)Y_V(L_V(-1)v_2, z_2)\mathbf{1} \rangle) \\ &= \langle w', E(\Psi(v_1, z_1)Y_V(L_V(-1)v_2, z_2)\mathbf{1}) \rangle \\ &= \langle w', (\Phi(v_1 \otimes L_V(-1)v_2))(z_1, z_2) \rangle. \end{aligned} \quad (3.13)$$

Using (3.8), (3.9) and the  $L(-1)$ -derivative property for  $V$ , we obtain

$$\left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2}\right) \langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2) \rangle$$

$$\begin{aligned}
&= \left( \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2} \right) \langle w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\
&= \left( \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2} \right) R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
&= R \left( \left\langle w', \frac{\partial}{\partial z_1} \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \right\rangle \right) \\
&\quad + R \left( \left\langle w', \Psi(v_1, z_1) \frac{\partial}{\partial z_2} Y_V(v_2, z_2)\mathbf{1} \right\rangle \right) \\
&= R(\langle w', \Psi(L_V(-1)v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
&\quad + R(\langle w', \Psi(v_1, z_1)Y_V(L_V(-1)v_2, z_2)\mathbf{1} \rangle) \\
&= R(\langle w', L_W(-1)\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
&= R(\langle L_{W'}(1)w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
&= \langle L_{W'}(1)w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\
&= \langle w', L_W(-1)E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\
&= \langle w', L_W(-1)(\Phi(v_1 \otimes v_2))(z_1, z_2) \rangle
\end{aligned} \tag{3.14}$$

for  $v_1, v_2 \in V$  and  $w' \in W'$ , From (3.12), (3.13) and (3.14), we see that  $\Phi$  satisfies the  $L(-1)$ -derivative property.

Also for  $v_1, v_2 \in V$  and  $w' \in W'$ , by (3.11) and the  $L(0)$ -bracket formula for  $V$ , we have

$$\begin{aligned}
&\langle w', z^{L_{W'}(0)}(\Phi(v_1 \otimes v_2))(z_1, z_2) \rangle \\
&= \langle w', z^{L_{W'}(0)}E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\
&= \langle z^{L_{W'}(0)}w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \rangle \\
&= R(\langle z^{L_{W'}(0)}w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
&= R(\langle w', z^{L_{W'}(0)}\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\
&= R(\langle w', \Psi(z^{L_V(0)}v_1, z_1)Y_V(z^{L_V(0)}v_2, z_2)\mathbf{1} \rangle) \\
&= \langle w', E(\Psi(z^{L_V(0)}v_1, z_1)Y_V(z^{L_V(0)}v_2, z_2)\mathbf{1}) \rangle \\
&= \langle w', (\Phi(z^{L_V(0)}v_1 \otimes z^{L_V(0)}v_2))(z_1, z_2) \rangle,
\end{aligned}$$

that is,  $\Phi$  satisfies the  $L(0)$ -conjugation property.

Since  $V \oplus W$  is a grading-restricted vertex algebra, for  $v_1, v_2, v_3 \in V$  and  $w' \in W'$ , the series

$$\langle (0, w'), Y_{V \oplus W}((v_1, 0), z_1)Y_{V \oplus W}((v_2, 0), z_2)Y_{V \oplus W}((v_3, 0), z_3)(\mathbf{1}, 0) \rangle$$

and

$$\langle (0, w'), Y_{V \oplus W}(Y_{V \oplus W}((v_1, 0), z_1 - z_2)(v_2, 0), z_2)Y_{V \oplus W}((v_3, 0), z_3)(\mathbf{1}, 0) \rangle$$

are absolutely convergent in the regions given by  $|z_1| > |z_2| > |z_3| > 0$  and by  $|z_2| > |z_1 - z_2|, |z_3| > 0$  and  $|z_2 - z_3| > |z_1 - z_2|$ , respectively, to a same rational function with the only possible poles at  $z_1 = z_2, z_1 = z_3, z_2 = z_3$ . But by (3.2) and (3.3), these series are equal to

$$\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle$$

and

$$\begin{aligned} & \langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle \\ & + \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle, \end{aligned}$$

respectively, and are absolutely convergent to a same rational function which in our convention is equal to

$$R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle)$$

and

$$\begin{aligned} & R(\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle \\ & + \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1} \rangle). \end{aligned}$$

In particular, we have proved that  $\Phi \in \widehat{C}_{\frac{1}{2}}^2(V, W)$ .

Since by (3.4),

$$\begin{aligned} (\Phi(v_1 \otimes v_2))(z_1, z_2) &= E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \\ &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \\ &= (\Phi(v_2 \otimes v_1))(z_2, z_1) \\ &= (\sigma_{12}(\Phi(v_2 \otimes v_1)))(z_1, z_2) \end{aligned}$$

for  $v_1, v_2 \in V$  and  $(z_1, z_2) \in F_2\mathbb{C}$ , that is,

$$\Phi(v_1 \otimes v_2) - \sigma_{12}(\Phi(v_2 \otimes v_1)) = 0$$

for  $v_1, v_2 \in V$ , We obtain

$$\begin{aligned} & \sum_{\sigma \in J_{2;1}} (-1)^{|\sigma|} \sigma(\Phi(v_1 \otimes v_2)) \\ &= \Phi(v_1 \otimes v_2) - \sigma_{12}(\Phi(v_2 \otimes v_1)) \\ &= 0 \end{aligned}$$

for  $v_1, v_2 \in V$ . So  $\Phi \in C^2(V, W)$ .

Next we show that  $\delta_{\frac{1}{2}}^2(\Phi) = 0$ . For  $v_1, v_2, v_3 \in V$ ,  $w' \in W'$ ,

$$\begin{aligned} & \langle w', ((\delta_{\frac{1}{2}}^2(\Phi))(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3) \rangle \\ &= R(\langle w', (E_W^{(1)}(v_1; \Phi(v_2 \otimes v_3)))(z_1, z_2, z_3) \rangle \\ & \quad + \langle w', (\Phi(v_1 \otimes E^{(2)}(v_2 \otimes v_3; \mathbf{1}))(z_1, z_2, z_3) \rangle) \\ & \quad - R(\langle w', (\Phi(E^{(2)}(v_1 \otimes v_2; \mathbf{1}) \otimes v_3))(z_1, z_2, z_3) \rangle \\ & \quad + \langle w', (E_{WV}^{W;(1)}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3) \rangle) \\ &= R(\langle w', Y_W(v_1, z_1) \Psi(v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle \\ & \quad + \langle w', \Psi(v_1, z_1) Y_V(v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle) \\ & \quad - R(\langle w', \Psi(Y_V(v_1, z_1 - z_2) v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle \\ & \quad + \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2) v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle). \end{aligned} \quad (3.15)$$

Since  $V \oplus W$  is a grading-restricted vertex algebra, we have the associativity property

$$\begin{aligned} & R(\langle (0, w'), Y_{V \oplus W}((v_1, 0), z_1) Y_{V \oplus W}((v_2, 0), z_2) Y_{V \oplus W}((v_3, 0), z_3) (\mathbf{1}, 0) \rangle) \\ &= R(\langle (0, w'), Y_{V \oplus W}(Y_{V \oplus W}((v_1, 0), z_1 - z_2) (v_2, 0), z_2) \cdot \\ & \quad \cdot Y_{V \oplus W}((v_3, 0), z_3) (\mathbf{1}, 0) \rangle), \end{aligned}$$

which, by (3.2) and (3.3), is equivalent to

$$\begin{aligned} & R(\langle w', \Psi(v_1, z_1) Y_V(v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle \\ & \quad + \langle w', Y_W(v_1, z_1) \Psi(v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle) \\ &= R(\langle w', \Psi(Y_V(v_1, z_1 - z_2) v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle \\ & \quad + \langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2) v_2, z_2) Y_V(v_3, z_3) \mathbf{1} \rangle), \end{aligned}$$

as we have noticed above. So the right-hand side of (3.15) is 0. Thus  $\Phi + \delta_2^1 C_2^1(V, W)$  is an element of  $H_{\frac{1}{2}}^2(V, W)$ .

Conversely, given any element of  $H_{\frac{1}{2}}^2(V, W)$ , let  $\Phi \in C_{\frac{1}{2}}^2(V, W)$  be a representative of this element. Then for any  $v_1, v_2 \in V$ , there exists  $N$  such that for  $w' \in W'$ ,  $\langle w', (\Phi(v_1 \otimes v_2))(z, 0) \rangle$  is a rational function of  $z$  with the only possible pole at  $z = 0$  of order less than or equal to  $N$ . For  $v_1, v_2 \in V$ , let  $\Psi(v_1, x)v_2 \in W((x))$  be given by

$$\langle w', \Psi(v_1, x)v_2 \rangle|_{x=z} = \langle w', (\Phi(v_1 \otimes v_2))(z, 0) \rangle.$$

for  $z \in \mathbb{C}^\times$ . For  $v_1, v_2 \in V$ , define  $Y_{V \oplus W}(v_1, x)v_2$  using (3.2). So we obtain a vertex operator map  $Y_{V \oplus W}$ . Reversing the proof above, we see that  $V \oplus W$  equipped with the vertex operator map  $Y_{V \oplus W}$  and the vacuum  $(\mathbf{1}, 0)$  is a grading-restricted vertex algebra and together with the projection  $p_1 : V \oplus W \rightarrow V$  and the embedding  $i_2 : W \rightarrow V \oplus W$ ,  $V \oplus W$  is a square-zero extension of  $V$  by  $W$ .

Next we prove that two elements of  $\ker \delta_{\frac{1}{2}}^2$  obtained this way are differed by an element of  $\delta_1 C^1(V, W)$  if and only if the corresponding square-zero extensions of  $V$  by  $W$  are equivalent.

Let  $\Phi_1, \Phi_2 \in \ker \delta_{\frac{1}{2}}^2$  be two such elements obtained from square-zero extensions  $(V \oplus W, Y_{V \oplus W}^{(1)}, p_1, i_2)$  and  $(V \oplus W, Y_{V \oplus W}^{(2)}, p_1, i_2)$ . Assume that  $\Phi_1 = \Phi_2 + \delta_1(\Gamma)$  where  $\Gamma \in C^1(V, W)$ . Since

$$\begin{aligned} & \langle w', ((\delta_1(\Gamma))(v_1 \otimes v_2))(z_1, z_2) \rangle \\ &= R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2) \rangle) \\ & \quad - R(\langle w', (\Gamma(Y_V(v_1, z_1 - z_2)v_2))(z_2) \rangle) \\ & \quad + R(\langle w', Y_W(v_2, z_2)(\Gamma(v_1))(z_1) \rangle), \end{aligned}$$

we have

$$\begin{aligned} & R(\langle w', \Psi_1(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\ &= \langle w', (\Phi_1(v_1 \otimes v_2))(z_1, z_2) \rangle \\ &= \langle w', (\Phi_2(v_1 \otimes v_2))(z_1, z_2) \rangle \\ & \quad + \langle w', (\delta_1(\Gamma))(z_1, z_2) \rangle \\ &= R(\langle w', \Psi_2(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle) \\ & \quad + R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2) \rangle) \\ & \quad - R(\langle w', (\Gamma(Y_V(v_1, z_1 - z_2)v_2))(z_2) \rangle) \\ & \quad + R(\langle w', Y_W(v_2, z_2)(\Gamma(v_1))(z_1) \rangle) \end{aligned}$$

$$\begin{aligned}
&= R(\langle w', \Psi_2(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} \rangle \\
&\quad + R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2) \rangle) \\
&\quad - R(\langle w', (\Gamma(Y_V(v_1, z_1)v_2))(z_2) \rangle) \\
&\quad + R(\langle w', e^{(z_1-z_2)L_W(-1)}Y_W(v_2, -z_1)(\Gamma(v_1))(z_2) \rangle). \tag{3.16}
\end{aligned}$$

Let  $z_2$  go to zero on both sides of (3.16). We obtain

$$\begin{aligned}
\langle w', \Psi_1(v_1, z_1)v_2 \rangle &= \langle w', \Psi_2(v_1, z_1)v_2 \rangle \\
&\quad + \langle w', Y_W(v_1, z_1)(\Gamma(v_2))(0) \rangle \\
&\quad - \langle w', (\Gamma(Y_V(v_1, z_1)v_2))(0) \rangle \\
&\quad + \langle w', e^{z_1 L_W(-1)}Y_W(v_2, -z_1)(\Gamma(v_1))(0) \rangle \\
&= \langle w', \Psi_2(v_1, z_1)v_2 \rangle \\
&\quad + \langle w', Y_W(v_1, z_1)(\Gamma(v_2))(0) \rangle \\
&\quad - \langle w', (\Gamma(Y_V(v_1, z_1)v_2))(0) \rangle \\
&\quad + \langle w', Y_{WV}^W((\Gamma(v_1))(0), z_1)v_2 \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\Psi_1(v_1, x)v_2 &= \Psi_2(v_1, x)v_2 + Y_W(v_1, x)(\Gamma(v_2))(0) \\
&\quad - (\Gamma(Y_V(v_1, x)v_2))(0) + Y_{WV}^W((\Gamma(v_1))(0), x)v_2. \tag{3.17}
\end{aligned}$$

For  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ , by (3.2) and (3.17), we have

$$\begin{aligned}
&Y_{V \oplus W}^{(1)}((v_1, w_1), x)(v_2, w_2) \\
&= (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi_1(v_1, x)v_2) \\
&= (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi_2(v_1, x)v_2) \\
&\quad + (Y_V(v_1, x)v_2, Y_W(v_1, x)(\Gamma(v_2))(0)) \\
&\quad - (Y_V(v_1, x)v_2, (\Gamma(Y_V(v_1, x)v_2))(0)) \\
&\quad + (Y_V(v_1, x)v_2, Y_{WV}^W((\Gamma(v_1))(0), x)v_2) \\
&= Y_{V \oplus W}^{(2)}((v_1, w_1 + (\Gamma(v_1))(0)), x)(v_2, w_2 + (\Gamma(v_2))(0)) \\
&\quad - (Y_V(v_1, x)v_2, (\Gamma(Y_V(v_1, x)v_2))(0)). \tag{3.18}
\end{aligned}$$

We now define a linear map  $h : V \oplus W \rightarrow V \oplus W$  by

$$e(v, w) = (v, w + (\Gamma(v))(0))$$

for  $v \in V$  and  $w \in W$ . Then  $h$  is a linear isomorphism and (3.18) can be rewritten as

$$h(Y_{V \oplus W}^{(1)}((v_1, w_1), x)(v_2, w_2)) = Y_{V \oplus W}^{(2)}(h(v_1, w_1), x)h(v_2, w_2). \quad (3.19)$$

for  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ . Thus  $h$  is an isomorphism of grading-restricted vertex algebras from  $(V \oplus W, Y_{V \oplus W}^{(1)}, (\mathbf{1}, 0))$  to  $(V \oplus W, Y_{V \oplus W}^{(2)}, (\mathbf{1}, 0))$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V \longrightarrow 0 \\ & & \downarrow 1_W & & \downarrow h & & \downarrow 1_V \\ 0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V \longrightarrow 0 \end{array} \quad (3.20)$$

is commutative. Thus the two square-zero extensions of  $V$  by  $W$  are equivalent.

Conversely, let  $(V \oplus W, Y_{V+W}^{(1)}, p_1, i_2)$  and  $(V \oplus W, Y_{V+W}^{(2)}, p_1, i_2)$  be two equivalent square-zero extensions of  $V$  by  $W$ . So there exists an isomorphism  $h : V \oplus W \rightarrow V \oplus W$  of grading-restricted vertex algebras such that (3.20) is commutative. We have the following lemma which is also needed in the next section:

**Lemma 3.1.** *There exists a linear map  $g : V \rightarrow V$  such that*

$$h(v, w) = (v, w + g(v))$$

for  $v \in V$  and  $w \in W$ .

*Proof.* Let  $h(v, w) = (f(v, w), g(v, w))$  for  $v \in V$  and  $w \in W$ . Then by (3.20), we have  $f(v, w) = v$  and  $g(0, w) = w$ . Since  $h$  is linear, we have  $g(v, w) = g(v, 0) + g(0, w) = w + g(v, 0)$ . So  $h(v, w) = (v, w + g(v, 0))$ . Taking  $g(v)$  to be  $g(v, 0)$ , we see that the conclusion holds. ■

Let  $(\Gamma(v))(z_1) = e^{z_1 L_W(-1)} g(v) \in \overline{W}$ . Then  $\Gamma : V \rightarrow \widetilde{W}_{z_1}$  is an element of  $C_2^1(V, W)$ . By definition, we have  $g(v) = (\Gamma(v))(0)$  and  $h(v, w) = (v, w + (\Gamma(v))(0))$  for  $v \in V$  and  $w \in W$ . Let  $\Phi_1$  and  $\Phi_2$  be elements of  $\ker \delta_{\frac{1}{2}}^2$  obtained from  $(V \oplus W, Y_{V \oplus W}^{(1)}, p_1, i_2)$  and  $(V \oplus W, Y_{V \oplus W}^{(2)}, p_1, i_2)$ , respectively, and  $\Psi_1$  and  $\Psi_2$  the corresponding maps from  $V \otimes V$  to  $W((x))$ . Then since  $h$  is a homomorphism of grading-restricted vertex algebras, (3.19) holds for



$v_1, v_2 \in V$  and  $w_1, w_2 \in W$ . Thus the two sides of (3.18) are equal for  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ . So the two expressions in the middle of (3.18) are equal for  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ . Thus we have (3.17) for  $v_1, v_2 \in V$ . Formula (3.17) implies that the two sides of (3.16) are equal for  $v_1, v_2 \in V$ . Thus the middle expressions in (3.16) are all equal for  $v_1, v_2 \in V$ . In particular, we obtain  $\Phi_1 = \Phi_2 + \Gamma$ . So  $\Phi_1$  and  $\Phi_2$  are differed by an element of  $\delta_1 C^1(V, W)$ . ■

## 4 Square-zero extensions and first order deformations

In this section, we prove Theorem 1.7.

Let  $Y_t : V \otimes V \rightarrow V((x))$ ,  $t \in U$ , be a first order deformation of  $V$ . By definition, there exists

$$\begin{aligned} \Psi : V \otimes V &\rightarrow V((x)) \\ v_1 \otimes v_2 &\rightarrow \Psi(v_1, x)v_2 \end{aligned}$$

such that

$$Y_t(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi(v_1, x)v_2$$

for  $v_1, v_2 \in V$ . By definition,  $(V, Y_t, \mathbf{1})$  satisfies all the axioms for grading-restricted vertex algebras up to the first order in  $t$  and consequently have all properties of grading-restricted vertex algebras up to the first order in  $t$ .

The identity property for  $(V, Y_t, \mathbf{1})$  up to the first order in  $t$  gives

$$Y_V(\mathbf{1}, x)v + t\Psi(\mathbf{1}, x)v = v + O(t^2)$$

for  $v \in V$ . So we obtain

$$\Psi(\mathbf{1}, x)v = 0 \tag{4.1}$$

for  $v \in V$ . The creation property for  $(V, Y_t, \mathbf{1})$  up to the first order in  $t$  gives

$$\lim_{t \rightarrow 0} (Y_V(v, x) + t\Psi(v, x))\mathbf{1} = v + O(t^2)$$

for  $v \in V$ . Then we have

$$\lim_{t \rightarrow 0} \Psi(v, x)\mathbf{1} = 0 \tag{4.2}$$

for  $v \in V$ .

We have the following duality property up to the first order in  $t$ : For  $v_1, v_2, v_3 \in V$  and  $v' \in V'$ , the coefficients of  $t^0$  and  $t^1$  terms of

$$\begin{aligned} & \langle v', Y_t(v_1, z_1)Y_t(v_2, z_2)v_3 \rangle \\ & \langle v', Y_t(v_2, z_2)Y_t(v_1, z_1)v_3 \rangle \\ & \langle v', Y_t(Y_t(v_1, z_1 - z_2)v_2, z_2)v_3 \rangle \end{aligned}$$

are absolutely convergent in the regions  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$  and  $|z_2| > |z_1 - z_2| > 0$ , respectively, to common rational functions in  $z_1$  and  $z_2$  with the only possible poles at  $z_1, z_2 = 0$  and  $z_1 = z_2$ , or equivalently,

$$\langle v', (Y_V(v_1, z_1)\Psi(v_2, z_2) + \Psi(v_1, z_1)Y_V(v_2, z_2))v_3 \rangle \quad (4.3)$$

$$\langle v', (Y_V(v_2, z_2)\Psi(v_1, z_1) + \Psi(v_2, z_2)Y_V(v_1, z_1))v_3 \rangle \quad (4.4)$$

$$\langle v', (Y_V(\Psi(v_1, z_1 - z_2)v_2, z_2) + \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2))v_3 \rangle \quad (4.5)$$

are absolutely convergent in the regions  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$  and  $|z_2| > |z_1 - z_2| > 0$ , respectively, to a common rational function in  $z_1$  and  $z_2$  with the only possible poles at  $z_1, z_2 = 0$  and  $z_1 = z_2$ .

Let

$$\begin{aligned} Y_{V \oplus V} : (V \oplus V) \otimes (V \oplus V) & \rightarrow (V \oplus V)[[x, x^{-1}]] \\ (u_1, v_1) \otimes (u_2, v_2) & \mapsto Y_{V \oplus V}((u_1, v_1), x)(u_2, v_2) \end{aligned}$$

be given by

$$\begin{aligned} & Y_{V \oplus V}((u_1, v_1), x)(u_2, v_2) \\ & = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi(u_1, x)u_2) \end{aligned} \quad (4.6)$$

for  $u_1, u_2, v_1, v_2 \in V$ . By (4.6) and (4.1),

$$\begin{aligned} Y_{V \oplus V}((\mathbf{1}, 0), x)(u, v) & = (Y_V(\mathbf{1}, x)u, Y_V(\mathbf{1}, x)v + Y_V(0, x)u + \Psi(\mathbf{1}, x)u) \\ & = (u, v) \end{aligned}$$

for  $u, v \in V$ , that is,  $(V \oplus V, Y_{V \oplus V}, (\mathbf{1}, 0))$  has the identity property. By (4.6) and (4.2),

$$\begin{aligned} & \lim_{x \rightarrow 0} Y_{V \oplus V}((u, v), x)(\mathbf{1}, 0) \\ & = (\lim_{x \rightarrow 0} Y_V(u, x)\mathbf{1}, \lim_{x \rightarrow 0} Y_V(u, x)0 + \lim_{x \rightarrow 0} Y_V(v, x)\mathbf{1} + \Psi(u, x)\mathbf{1}) \\ & = (u, v) \end{aligned}$$

for  $u, v \in V$ , that is,  $(V \oplus V, Y_{V \oplus V}, (\mathbf{1}, 0))$  has the creation property.

By (4.6), we have

$$\begin{aligned}
& \langle (u', v'), Y_{V \oplus V}((u_1, v_1), z_1) Y_{V \oplus V}((u_2, v_2), z_2)(u_3, v_3) \rangle \\
&= \langle (u', v'), Y_{V \oplus V}((u_1, v_1), z_1) \cdot \\
&\quad \cdot (Y_V(u_2, z_2)u_3, Y_V(u_2, z_2)v_3 + Y_V(v_2, z_2)u_3 + \Psi(u_2, z_2)u_3) \rangle \\
&= \langle u', Y_V(u_1, z_1)Y_V(u_2, z_2)u_3 \rangle + \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v_3 \rangle \\
&\quad + \langle v', Y_V(u_1, z_1)Y_V(v_2, z_2)u_3 \rangle + \langle v', Y_V(u_1, z_1)\Psi(u_2, z_2)u_3 \rangle \\
&\quad + \langle v', Y_V(v_1, z_1)Y_V(u_2, z_2)u_3 \rangle + \langle v', \Psi(u_1, z_1)Y_V(u_2, z_2)u_3 \rangle. \tag{4.7}
\end{aligned}$$

By the properties of  $V$  and the absolute convergence of (4.3), we see that the left-hand side of (4.7) is absolutely convergent when  $|z_1| > |z_2| > 0$ . Similarly, by (4.6), we have

$$\begin{aligned}
& \langle (u', v'), Y_{V \oplus V}((u_2, v_2), z_2) Y_{V \oplus V}((u_1, v_1), z_1)(u_3, v_3) \rangle \\
&= \langle u', Y_V(u_2, z_2)Y_V(u_1, z_1)u_3 \rangle + \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v_3 \rangle \\
&\quad + \langle v', Y_V(u_2, z_2)Y_V(v_1, z_1)u_3 \rangle + \langle v', Y_V(u_2, z_2)\Psi(u_1, z_1)u_3 \rangle \\
&\quad + \langle v', Y_V(v_2, z_2)Y_V(u_1, z_1)u_3 \rangle + \langle v', \Psi(u_2, z_2)Y_V(u_1, z_1)u_3 \rangle \tag{4.8}
\end{aligned}$$

and the left-hand side of (4.8) is absolutely convergent when  $|z_2| > |z_1| > 0$ . Moreover, since (4.3) and (4.4) converges absolutely when  $|z_1| > |z_2| > 0$  and when  $|z_2| > |z_1| > 0$ , respectively, to a common rational function with the only possible poles at  $z_1, z_2, z_1 - z_2 = 0$ , the left-hand side of (4.7) and left-hand side of (4.8) also converges absolutely when  $|z_1| > |z_2| > 0$  and when  $|z_2| > |z_1| > 0$ , respectively, to a common rational function with the only possible pole at  $z_1 - z_2 = 0$ . By (4.6) again, we have

$$\begin{aligned}
& \langle (u', v'), Y_{V \oplus V}(Y_{V \oplus V}((u_1, v_1), z_1 - z_2)(u_2, v_2), z_2)(u_3, v_3) \rangle \\
&= \langle (u', v'), Y_{V \oplus V}((Y_V(u_1, z_1 - z_2)u_2, \\
&\quad Y_V(u_1, z_1 - z_2)v_2 + Y_V(v_1, z_1 - z_2)u_2 \\
&\quad + \Psi(u_1, z_1 - z_2)u_2), z_2)(u_3, v_3) \rangle \\
&= \langle u', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)u_3 \rangle + \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v_3 \rangle \\
&\quad + \langle v', Y_V(Y_V(u_1, z_1 - z_2)v_2, z_2)u_3 \rangle + \langle v', Y_V(Y_V(v_1, z_1 - z_2)u_2, z_2)u_3 \rangle \\
&\quad + \langle v', Y_V(\Psi(u_1, z_1 - z_2)u_2, z_2)u_3 \rangle + \langle v', \Psi(Y_V(u_1, z_1 - z_2)u_2, z_2)u_3 \rangle. \tag{4.9}
\end{aligned}$$

By the properties of  $V$  and the absolute convergence of (4.5) and (4.9), we see that the left-hand side of (4.9) is absolutely convergent when  $|z_2| > |z_1 - z_2| > 0$ . Moreover, since (4.3) and (4.5) converges absolutely when  $|z_1| > |z_2| > 0$  and when  $|z_2| > |z_1 - z_2| > 0$ , respectively, to a common rational function with the only possible poles at  $z_1, z_2, z_1 - z_2 = 0$ , the left-hand side of (4.7) and left-hand side of (4.9) also converges absolutely when  $|z_1| > |z_2| > 0$  and when  $|z_2| > |z_1 - z_2| > 0$ , respectively, to a common rational function with the only possible poles at  $z_1, z_2, z_1 - z_2 = 0$ . So  $(V \oplus V, Y_{V \oplus V}, (\mathbf{1}, 0))$  has the duality property.

Note that the  $L(-1)$ -derivative property and the  $L(-1)$ -bracket property are consequences of the other axioms. Thus  $(V \oplus V, Y_{V \oplus V}, (\mathbf{1}, 0))$  is a grading-restricted vertex algebra.

By definition,

$$\begin{aligned} p_1(Y_{V \oplus V}((u_1, v_1), x)(u_2, v_2)) \\ &= p_1(Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi(u_1, x)u_2) \\ &= Y_V(u_1, x)u_2 \\ &= Y_V(p_1(u_1, v_1), x)p_1(u_2, v_2) \end{aligned}$$

for  $u_1, u_2, v_1, v_2 \in V$ . Also

$$\ker p_1 = 0 \oplus V$$

and

$$Y_{V \oplus V}((0, v_1), x)(0, v_2) = (0, 0)$$

for  $v_1, v_2 \in V$ . So  $p_1$  is a surjective homomorphism of grading-restricted vertex algebras and  $\ker p_1$  is a square-zero ideal of  $V \oplus V$ .

We use  $Y_{V \oplus V}^V$  to denote the vertex operator map for  $V \oplus V$  when  $V \oplus V$  is viewed as a  $V$ -module. Then by definition,

$$\begin{aligned} i_2(Y_V(v_1, x)v_2) &= (0, Y_V(v_1, x)v_2) \\ &= Y_{V \oplus V}^V(v_1, x)(0, v_2) \\ &= Y_{V \oplus V}^V(v_1, x)i_2(v_2) \end{aligned}$$

for  $v_1, v_2 \in V$ . So  $i_2$  is an injective homomorphism of  $V$ -modules. Clearly, we have  $i_2(V) = \ker p_1$ . Thus  $(V \oplus V, Y_{V \oplus V}, p_1, i_2)$  is a square-zero extension of  $V$  by  $V$ .

Conversely, let  $(V \oplus V, Y_{V \oplus V}, p_1, i_2)$  be a square-zero extension of  $V$  by  $V$ . Then there exists

$$\begin{aligned}\Psi : V \otimes V &\rightarrow V((x)) \\ v_1 \otimes v_2 &\rightarrow \Psi(v_1, x)v_2\end{aligned}$$

such that

$$Y_{V \oplus V}((u_1, 0), x)(u_2, 0) = (Y_V(u_1, x)u_2, \Psi(u_1, x)u_2)$$

for  $u_1, u_2 \in V$ . The identity property and the creation property of the grading-restricted vertex algebra  $(V \oplus V, Y_{V \oplus V}, (\mathbf{1}, 0))$  give (4.1) and (4.2). The duality property for  $(V \oplus V, Y_{V \oplus V}, (\mathbf{1}, 0))$  gives (4.3), (4.4) and (4.5).

For  $t \in \mathbb{C}$ , define

$$Y_t(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi(v_1, x)v_2$$

for  $v_1, v_2 \in V$ . Then (4.1) and (4.2) imply that  $Y_t$  satisfies the identity property and the creation property up to the first order in  $t$  and (4.3), (4.4) and (4.5) imply that  $Y_t$  satisfies the duality property up to the first order in  $t$ . So  $(V, Y_t, \mathbf{1})$  for  $t \in \mathbb{C}$  is a family of grading-restricted vertex algebras up to the first order in  $t$  and thus  $Y_t$  is a first-order deformation of  $(V, Y_V, \mathbf{1})$ .

Now we prove that two first-order deformations of  $V$  are equivalent if and only if the corresponding square-zero extensions of  $V$  by  $V$  are equivalent.

Consider two equivalent first-order deformations of  $V$  given by  $Y_t^{(1)} : V \otimes V \rightarrow V((x))$  and  $Y_t^{(2)} : V \otimes V \rightarrow V((x))$  for  $t \in \mathbb{C}$ . Then there exist a family  $f_t : V \rightarrow V$ ,  $t \in \mathbb{C}$ , of linear maps of the form  $f_t = 1_V + tg$  where  $g : V \rightarrow V$  is a linear map preserving the grading of  $V$  such that (1.1) holds for  $v_1, v_2 \in V$ . By definition, there exist linear maps

$$\begin{aligned}\Psi_1 : V \otimes V &\rightarrow V((x)) \\ v_1 \otimes v_2 &\rightarrow \Psi_1(v_1, x)v_2\end{aligned}$$

and

$$\begin{aligned}\Psi_2 : V \otimes V &\rightarrow V((x)) \\ v_1 \otimes v_2 &\rightarrow \Psi_2(v_1, x)v_2\end{aligned}$$

such that  $Y_t^{(1)} = Y_V + t\Psi_1$  and  $Y_t^{(2)} = Y_V + t\Psi_2$ . By (1.1), we have

$$\begin{aligned}\Psi_1(v_1, x)v_2 - \Psi_2(v_1, x)v_2 \\ = -g(Y_V(v_1, x)v_2) + Y_V(g(v_1), x)v_2 + Y_V(v_1, x)g(v_2)\end{aligned}\quad (4.10)$$

for  $v_1, v_2 \in V$ .

Let  $(V \oplus V, Y_{V \oplus V}^{(1)}, p_1, i_2)$  and  $(V \oplus V, Y_{V \oplus V}^{(2)}, p_1, i_2)$  be the square-zero extensions of  $V$  by  $V$  constructed from  $Y_t^{(1)}$  and  $Y_t^{(2)}$ . Let  $h : V \oplus V \rightarrow V \oplus V$  be defined by

$$h(v_1, v_2) = (v_1, v_2 + g(v_1))$$

for  $v_1, v_2 \in V$ . Clearly,  $h$  is a linear isomorphism. For  $u_1, u_2, v_1, v_2 \in V$ , by definition and (4.10),

$$\begin{aligned} & h(Y_{V \oplus V}^{(1)}((u_1, v_1), x)(u_2, v_2)) \\ &= h(Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi_1(u_1, x)u_2) \\ &= (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 \\ &\quad + \Psi_1(u_1, x)u_2 + g(Y_V(u_1, x)u_2)) \\ &= (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 \\ &\quad + \Psi_2(u_1, x)u_2 + Y_V(g(u_1), x)u_2 + Y_V(u_1, x)g(u_2)) \\ &= (Y_V(u_1, x)u_2, Y_V(u_1, x)(v_2 + g(u_2)) \\ &\quad + Y_V((v_1 + g(u_1)), x)u_2 + \Psi_2(u_1, x)u_2) \\ &= Y_{V \oplus V}^{(2)}(h(u_1, v_1), x)h(u_2, v_2). \end{aligned}$$

So  $h$  is in fact an isomorphism from the algebra  $(V \oplus V, Y_{V \oplus V}^{(1)}, (\mathbf{1}, 0))$  to the algebra  $(V \oplus V, Y_{V \oplus V}^{(2)}, (\mathbf{1}, 0))$ . Now it is clear that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i_2} & V \oplus V & \xrightarrow{p_1} & V & \longrightarrow & 0 \\ & & \downarrow 1_W & & \downarrow h & & \downarrow 1_V & & \\ 0 & \longrightarrow & V & \xrightarrow{i_2} & V \oplus V & \xrightarrow{p_1} & V & \longrightarrow & 0, \end{array}$$

So these two first order deformations are equivalent.

Conversely, let  $(V \oplus V, Y_{V \oplus V}^{(1)}, p_1, i_2)$  and  $(V \oplus V, Y_{V \oplus V}^{(2)}, p_1, i_2)$  be two equivalent square-zero extensions of  $V$  by  $V$ . Let  $\Psi_1, \Psi_2 : V \otimes V \rightarrow V((x))$  be given by

$$\begin{aligned} Y_{V \oplus V}^{(1)}((u_1, 0), x)(u_2, 0) &= (Y_V(u_1, x)u_2, \Psi_1(u_1, x)u_2), \\ Y_{V \oplus V}^{(2)}((u_1, 0), x)(u_2, 0) &= (Y_V(u_1, x)u_2, \Psi_2(u_1, x)u_2) \end{aligned}$$

for  $u_1, u_2 \in V$ . Then  $Y_t^{(1)}, Y_t^{(2)} : V \otimes V \rightarrow V((x))$  given by

$$\begin{aligned} Y_t^{(1)}(v_1, x)v_2 &= Y_V(v_1, x)v_2 + t\Psi_1(v_1, x)v_2, \\ Y_t^{(2)}(v_1, x)v_2 &= Y_V(v_1, x)v_2 + t\Psi_2(v_1, x)v_2 \end{aligned}$$

for  $v_1, v_2 \in V$  are first-order deformations of  $(V, Y_V, \mathbf{1})$  by the proof above.

Let  $h : V \oplus V \rightarrow V \oplus V$  be an equivalence from  $(V \oplus V, Y_{V \oplus V}^{(1)}, p_1, i_2)$  to  $(V \oplus V, Y_{V \oplus V}^{(2)}, p_1, i_2)$ . Then by Lemma 3.1, there exists a linear map  $g : V \rightarrow V$  such that

$$h(v_1, v_2) = (v_1, v_2 + g(v_1))$$

for  $v_1, v_2 \in V$ . Using the fact that  $h$  is an isomorphism of grading-restricted vertex algebras from  $(V \oplus V, Y_{V \oplus V}^{(1)}, (\mathbf{1}, 0))$  to  $(V \oplus V, Y_{V \oplus V}^{(2)}, (\mathbf{1}, 0))$ , we obtain (4.10) which implies (1.1). Thus the two first-order deformations  $Y_t^{(1)}$  and  $Y_t^{(2)}$  are equivalent.  $\blacksquare$

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